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Isospectral graphs and the representation-theoretical spectrum

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Abstract

A finite connected k-regular graph $X, k \geq 3$, determines the conjugacy class of a cocompact torsion-free lattice Γ in the isometry group G of the universal covering tree. The associated quasi-regular representation $L^2(\Gamma \backslash G)$ of G can be considered as an a priori stronger notion of the spectrum of X, called the representation spectrum. We prove that two graphs as above are isospectral if and only if they are representation-isospectral. In other words, for a cocompact torsion-free lattice Γ in G the spherical part of the spectrum of Γ determines the whole spectrum. We give examples to show that this is not the case if the lattice has torsion.

0. Introduction

Let $k \geq 3$ be an integer, X be a k-regular graph, $A = A_X$ its adjacency operator of X, i.e., A: $L^2(X) \longrightarrow L^2(X)$, with

$$A(f)(x) = \sum_{y \sim x} f(y),$$

where $y \sim x$ means that y is a neighbor of x. This is a self-adjoint operator. Two finite k-regular graphs Y_1 and Y_2 are said to be *isospectral* (or *cospectral*) if the sets of eigenvalues of A_{Y_i} (with multiplicities) are equal to each other. Such a Y_i determines (the conjugacy class of) a cocompact lattice (discrete cocompact subgroup) Γ_i of $G = \mathrm{Aut}(T)$ where T is

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the k-regular tree such that $\Gamma_i \backslash G = Y_i$. Given a cocompact lattice Γ the group G acts on $L^2(\Gamma\backslash G)$ and this unitary representation is decomposed as

$$\bigoplus_{\pi\in\hat{G}} m_\Gamma(\pi)\pi,$$

where $\hat{G}$ is the unitary dual of G (i.e., the set of equivalence classes of irreducible unitary representations of G). Each $m_\Gamma(\pi) \in \mathbb{N} = \{0, 1, 2, \ldots\}$ and $m_\Gamma(\pi) \neq 0$ for at most countably many π's in $\hat{G}$. Let $\hat{G}_{\text{sph}}$ be the subset of $\hat{G}$ of all $\pi \in \hat{G}$ which are spherical (i.e., which have a non-trivial fixed vector under the stabilizer K of a vertex t in T; K is a maximal compact subgroup of G). It is well known that two graphs Y_1 and Y_2 are isospectral iff $m_{\Gamma_1}(\pi) = m_{\Gamma_2}(\pi)$ for each $\pi \in \hat{G}_{\text{sph}}$. We can define an a priori stronger equivalence relation: say that Y_1 and Y_2 are *representation-isospectral* if $m_{\Gamma_1}(\pi) = m_{\Gamma_2}(\pi)$ for every $\pi \in \hat{G}$. The main goal of this note is to put on record the observation that this is not a stronger relation.

Theorem. *Two finite k-regular graphs are isospectral iff they are representation-isospectral.*

Remark 1. The notions of isospectrality and representation-isospectrality makes equally good sense for cocompact lattices and we use the same terminology for such lattices.

We also present examples to show that this is not the case for general cocompact lattices. That is, we present two cocompact lattices Γ_1 and Γ_2 of G such that the quotients $\Gamma_1\backslash T$ and $\Gamma_2\backslash T$ are isomorphic but Γ_1 and Γ_2 are not representation-isospectral. These Γ_i have torsion, so they cover "indexed diagrams" but not graphs.

The paper is organized as follows. In Sections 1 and 2 we give two proofs of the theorem above. In Section 3 we present the promised Γ_1 and Γ_2. In Section 4 we discuss briefly the analogous question when G is replaced by a semisimple Lie group. In fact, Pesce [2] proved a similar result for Riemann surfaces (i.e., for $SL_2(\mathbb{R})$) and we show that the first proof works for $SL_2(K)$ when K is a non-Archimedean local field. The general case seems to be an interesting open question.

1. First proof

Assume that Y_1 and Y_2 are two isospectral graphs. Let Γ_1 and Γ_2 be their respective fundamental groups. Consider, as before, the unitary representations $L^2(\Gamma_i\backslash G)$ of G for $i = 1, 2$. Denote by R_i the corresponding quasi-regular representation of G, i.e.,

$$R_i(y)(f)(x) := f(xy)$$

for each $x, y \in G$ and each $f \in L^2(\Gamma_i\backslash G)$. If $\mathbf{H}(G)$ denotes the convolution algebra of complex locally constant functions with compact support on G, then the representations R_i give rise to representations of $\mathbf{H}(G)$ which we denote again by R_i. There are two ingredients in the proof. One of them is an elementary version of the Selberg trace formula which reads as follows ([9]). Let Γ be a cocompact lattice in G and let R denote the corresponding quasi-regular representation of G on $L^2(\Gamma\backslash G)$. Then

$$\mathrm{tr}(R(f)) = \sum_{\pi\in\hat{G}} m(\pi)\mathrm{tr}(\pi(f)) = \sum_{\gamma\in\{\Gamma\}} \mathrm{vol}(\Gamma_\gamma\backslash G_\gamma)\mathcal{O}_\gamma(f),$$

where $f \in \mathbf{H}(G)$, $\{\Gamma\}$ is a representative set for the Γ-conjugacy classes of elements in Γ, and $\mathcal{O}_\gamma(f)$ means $\int_{G_\gamma\backslash G} f(y^{-1}\gamma y)\mathrm{d}y$.

Remark 2. Let us note that the groups G_γ are also unimodular and we can fix any Haar measures on them. Then we consider the unique G-invariant measure on $G_\gamma\backslash G$ corresponding to it which satisfies some natural properties. The trace of the operator $R(f)$ does not depend on the choice of the Haar measure on G_γ. The reader is referred to [9, Theorem 5.9] for the details.

The other ingredient is a result from representation theory which gives a criterion for the equivalence of these unitary representations.

Proposition 3 ([4]). *The representation R_1 and R_2 are equivalent iff, for any $f \in \mathbf{H}(G)$, we have* $\mathrm{tr}(R_1(f)) = \mathrm{tr}(R_2(f))$.

Now let Γ be a cocompact torsion-free lattice in G. If an element $\gamma \in \Gamma$ is not the identity, then it is hyperbolic, i.e., there is a doubly infinite geodesic, which we call the axis of γ, on which γ acts as a translation. It is known that two such elements are G-conjugate iff they have the same translation length on their axes. Therefore we can partition Γ first into G-conjugacy classes and then into Γ-conjugacy classes and can rewrite the right hand side of the trace formula as

$$\sum_{n=0}^{\infty}\sum_{\gamma\in\{\Gamma\}_n} \mathrm{vol}(\Gamma_\gamma\backslash G_\gamma)\mathcal{O}_\gamma(f) = \sum_{n=0}^{\infty}\mathcal{O}_n(f)\sum_{\gamma\in\{\Gamma\}_n} \mathrm{vol}(\Gamma_\gamma\backslash G_\gamma)$$

where $\{\Gamma\}_n$ denotes the set of Γ-conjugacy classes in Γ of an element of translation length n. Remark that $\mathcal{O}_\gamma(f)$ is an integral over the G-conjugacy class of γ in G and it depends only on the translation length n of γ. In other words, if γ and γ' have the same translation length n, then $\mathcal{O}_\gamma(f) = \mathcal{O}_{\gamma'}(f)$ which we denote by $\mathcal{O}_n(f)$. Therefore, if we define $\alpha(n) = \alpha_\Gamma(n) = \sum_{\gamma\in\{\Gamma\}_n}\mathrm{vol}(\Gamma_\gamma\backslash G_\gamma)$, these quantities determine $\mathrm{tr}(R(f))$ uniquely. Since we want to prove that the representation R is uniquely determined (up to unitary equivalence, of course) by the spectrum of the quotient X of T by Γ, it is enough to prove that the function α is uniquely determined by the spectrum of X. Let us see first that the function $\alpha(n)$ depends only on the numbers of primitive conjugacy classes of degree $\leq n$. Now let γ be a hyperbolic element which is primitive. This means that Γ_γ is the infinite cyclic subgroup of Γ (and hence of G) generated by γ. Let l be the degree of γ considered as a translation on its axis. Now, up to G-conjugacy G_γ depends only on l, since two such elements are G-conjugate iff their degrees are the same. Let us denote the centralizer of a hyperbolic element γ of degree l by G_l. By the same token, $\mathrm{vol}(\Gamma_\gamma\backslash G_\gamma) = \mathrm{vol}(\langle\gamma\rangle\backslash G_l)$ is a function of l only.

In general, the centralizer of γ in Γ is free cyclic and it is generated by a unique primitive element. Call this element γ_0. Then $(\Gamma)_\gamma = \langle\gamma_0\rangle$ and the volume $\mathrm{vol}(\Gamma_\gamma\backslash G_\gamma)$ is a function of the lengths of γ and γ_0 only. In other words, if $p(\gamma)$ is the integer which satisfies $(\gamma_0)^{p(\gamma)} = \gamma$, then $\mathrm{vol}((\Gamma)_\gamma \backslash G_\gamma)$ is a function $\phi(l(\gamma), p(\gamma))$ of $l(\gamma)$ and $p(\gamma)$. Remark that a non-trivial element γ of Γ is primitive if, and only if, $p(\gamma) = 1$. We have then

$$\alpha(n) = \sum_{d|n} \beta(d)\phi(n, n/d)$$

where $\beta(d)$ denotes the number of primitive Γ-conjugacy classes in Γ of degree d.

It is known that two k-regular graphs are isospectral iff, for each l, the numbers of primitive conjugacy classes of degree l are the same [8]. But this means that, if two finite k-regular graphs X and Y are isospectral, then the corresponding α_X and α_Y are the same. Hence they are representation-isospectral.

2. Second proof

The second proof which we now present is based on the following observations. According to the classification of Ol'shanskii [6], the irreducible (unitary) representations of G are either spherical (i.e., have a non-zero vector fixed under the stabilizer of a point), special (i.e., have a non-zero vector fixed under the stabilizer of an oriented edge, which is called the Iwahori subgroup), or cuspidal (i.e., do not have any non-zero vector fixed under the Iwahori subgroup). Irreducible cuspidal (smooth) representations also have the property that they are integrable discrete series representations and their smooth parts have compactly supported matrix coefficients. Let Γ be a cocompact torsion-free lattice in G. We are interested in m_Γ. Ihara [3] proved that the multiplicities of the special representations in $L^2(\Gamma \backslash G)$ are uniquely determined by the genus, and hence by the size, of the quotient graph. Therefore, in order to show that isospectrality implies representation-isospectrality, it suffices to show that the multiplicities of irreducible cuspidal representations are determined by the spectrum. We prove an even stronger result: we show that, if π is an irreducible cuspidal G-module, then $m_\Gamma(\pi)$ is equal to $d_\pi \mathrm{vol}(\Gamma \backslash G)$, where d_π is the formal degree of π. This is an analog of a well-known result in the theory of semisimple Lie groups [9]. For let (π, V) be an irreducible cuspidal G-module. Let v be a smooth unit vector in V. Write

$$f(x) = d_\pi \langle \pi(x^{-1})(v), v \rangle$$

for each $x \in G$. Then, it is known that f is an idempotent element of $\mathbf{H}(G)$ with $\mathrm{tr}(\pi(f)) = 1$ and that, for any other irreducible smooth G-module π', we have $\pi'(f) = 0$. This follows from the definitions and Schur's orthogonality relations for discrete series representations. Therefore, in the notation of the introduction, we have

$$\mathrm{tr}(R(f)) = m_\Gamma(\pi) = \sum_{n=0}^{\infty} \mathcal{O}_n(f) \sum_{\gamma \in \{\Gamma\}_n} \mathrm{vol}(\Gamma_\gamma \backslash G_\gamma).$$

Now we use a result of Julg and Valette, which is known as the Selberg Principle [5]. It says that, if $f \in \mathbf{H}(G)$ is idempotent and $\gamma \in G$ is a hyperbolic element, then $\mathcal{O}_\gamma(f) = 0$. As is easily seen, this principle says that we have

$$\mathcal{O}_n(f) = 0$$

for all $n > 0$. Therefore, we have

$$m_\Gamma(\pi) = \mathcal{O}_0(f)\,\mathrm{vol}(\Gamma\backslash G).$$

But it is clear that $\mathcal{O}_0(f) = d_\pi$. Hence the result follows.

3. The cases where the lattices may have torsion

In the case of regular trees of even degree > 3, Bass and Kulkarni proved that one can find an infinite ascending chain of lattices with the same quotient [1, Example 7.4] containing only one vertex. Therefore, we can always find two cocompact lattices Γ_1 and Γ_2 with the same spherical spectra (the spherical parts of the corresponding quasi-regular representations of G are equivalent; there is only the trivial representation with multiplicity one) such that $\Gamma_1 < \Gamma_2$ and Γ_2/Γ_1 is as large as we want. But, then, it is not difficult to see that the unitary G-representations $L^2(\Gamma_1\backslash G)$ and $L^2(\Gamma_2\backslash G)$ of G will not be isomorphic. Hence these lattices are not representation-isospectral.

4. The case of semisimple Lie groups

Let us remark that the above result can be extended to some closed subgroups of G. We are now interested in the case where T is the Bruhat–Tits building associated with the p-adic Lie group $SL(2,\mathbb{Q}_p)$ and that Γ is a cocompact torsion-free lattice in $SL(2,\mathbb{Q}_p)$ with $X = \Gamma\backslash T$. It follows from the (first) proof that it is enough to observe that, for each non-trivial $\gamma \in \Gamma$ (which is automatically hyperbolic), $\mathrm{vol}(\Gamma_\gamma\backslash SL(2,\mathbb{Q}_p)_\gamma)$ depends only on the hyperbolic length of γ. Therefore, two cocompact torsion-free lattices in $SL(2,\mathbb{Q}_p)$ are (spherically) isospectral iff they are representation-isospectral.

It is natural to ask the same question for semisimple Lie groups and their cocompact lattices. For example, it would be interesting to understand the same question for cocompact torsion-free lattices in general semisimple Lie groups of rank one and their p-adic analogs. Pesce [2] proved that two compact hyperbolic manifolds are representation-isospectral iff they are strongly isospectral. This last condition says simply that one is not interested only in the Laplacian on functions, but also in some other Laplacians on forms, tensors, etc. It is still not clear whether representation-isospectrality is strictly stronger than isospectrality in this case. In fact, Pesce proved in [2] that isospectrality and representation-isospectrality are equivalent in the case of $PSL(2,\mathbb{R})$. (In the case of hyperbolic Riemann surfaces the notion of strong isospectrality reduces to that of isospectrality.) In other words, he proved that two compact Riemann surfaces are isospectral iff they are representation-isospectral in the sense of this note. Actually the result presented in this note is the graph-theoretical analog of his.

For higher rank semisimple Lie groups much less is known. There are some examples where non-isomorphic but isospectral (torsion-free cocompact) lattices exist. See, for example [7]. It is not clear whether there exist examples of higher rank Lie groups where (spherical) isospectrality and representation-isospectrality of cocompact torsion-free lattices are equivalent.

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